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The hidden angular momenta of Racah and $3n$-j coefficients

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Abstract. The Racah formula for the SU(2) 6-j coefficient is usually considered as a pure combinatorial formula. A physical interpretation is found for this formula and, more generally, for combinatorial formulae of the $3n$-j coefficients. Angular momenta are associated with the p_n points of the finite projective geometry PG$(n, 2)$ where triangular conditions appear as collinearities of points. A $3n$-j coefficient corresponds to a subset of PG$(n, 2)$ so that some of the p_n angular momenta are hidden for the $3n$-j. The combinatorial formula of the $3n$-j is interpreted as the summation over these hidden angular momenta of a highly symmetric 'full p_n-J symbol'.

1. Introduction

Many links between geometry and the theory of angular momentum in quantum mechanics have long been known (Wigner [1], Fano and Racah [2], Biedenharn and Louck [3]). When coupling two angular momenta j_1 and j_2 to form j_3, we often think of the triplet (j_1, j_2, j_3) as a triangle with sides of length j_1, j_2 and j_3 and speak of *triangular condition* (euclidian geometry). In the *angular momentum graphs* introduced by Jucys (whose name is also spelled Yutsis), Levinson and Vanagas [4, 5] the momenta are associated with edges and triangular conditions with vertices (graph theory). The association of momenta to faces of tetrahedra has been considered more recently by Barbieri [6] (simplicial geometry). In this paper momenta will be associated with points and triangular conditions with collinearities of points (projective geometry).

In Fano and Racah [2], the 6-j coefficient $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ is depicted by a complete quadrilateral (figure 1), where the four triangular conditions correspond to the four lines. This description has been used to interpret various relations between 6-j coefficients [2, 3]. Robinson [7], remarking that we are really interested in only three points of each line, introduced the finite projective geometries PG$(n, 2)$.

The finite projective space PG$(2, 2)$, also known as the Fano plane (figure 2), contains seven points and seven lines. Each line contains exactly three points (one line is depicted as a circle in figure 2). PG$(2, 2)$ is constructed from the three-dimensional vector space $\mathbb{F}_2^3$ over the two-element field $\mathbb{F}_2 = \{0, 1\}$. The space $\mathbb{F}_2^3$ is formed of eight elements of coordinates (λ, μ, ν) in the standard basis which we code by the binary number $\lambda\mu\nu$. The seven non-zero elements of $\mathbb{F}_2^3$ are identified with the points of PG$(2, 2)$. Given two different points e and e' of PG$(2, 2)$, there is exactly one line passing through these points and the third point e'' on the line corresponds to relation $e + e' + e'' = 0$ in $\mathbb{F}_2^3$. By placing the

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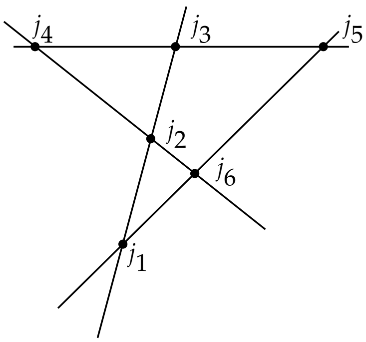

Figure 1. The 6-j coefficient in projective geometry.

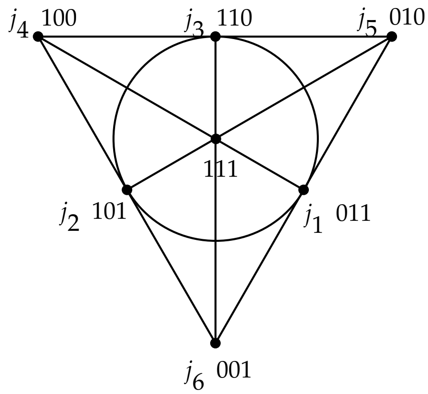

Figure 2. The Fano plane PG(2, 2) of momenta for the 6-j.

quadrilateral (figure 1) in figure 2, Robinson interpreted six of the seven points of PG(2, 2) as a 6-j symbol.

In this paper we present an interpretation of the 6-j coefficient in terms of *all seven* points of PG(2, 2). The momenta of the 6-j are associated with six points of PG(2, 2) as in Robinson and a *hidden momentum* is associated with the seventh point (point 111 in figure 2), thus forming a set of seven momenta and a *full 7-J symbol*. The hidden momentum can take any value satisfying the triangular conditions represented by the three lines passing through point 111. The dual projective space PG(2^*, 2) is a second Fano plane. Points in one Fano plane are in duality relation with lines in the other Fano plane, collinear points corresponding to concurrent lines. We associate one non-negative integer to each of the seven points of PG(2^*, 2) that we call the *co-momentum* of that point. The seven co-momenta are discrete Fourier transforms of the seven momenta. We define the value of the full 7-J symbol as a simple expression which is invariant in all 7! permutations of the co-momenta. The Racah formula for the 6-j coefficient is then interpreted as the summation of the full 7-J symbol over all values of the hidden momentum.

We have obtained similar results for the general 3n-j coefficient, summarized by the diagram:

$$\boxed{\text{co-momenta}} \quad \overset{\text{duality}}{\longleftrightarrow} \quad \boxed{\begin{array}{c}\boxed{\text{momenta}}\\ \boxed{\text{hidden momenta}}\end{array}} \qquad (1)$$

As another example, the geometry of the 9-j coefficient is the three-dimensional 15-point space PG(3, 2) (figure 3). Adding six hidden momenta to the 9-j we define a full 15-J symbol with 15! permutational symmetries in the space of co-momenta. The value of the 9-j is obtained by a summation over the six hidden momenta of this full 15-J symbol multiplied by a phase factor.

We begin by defining the 3n-j coefficient from its angular momentum graph G (section 2). The cycles of G (section 3) are used to define the spaces PG(n^*, 2) (section 4) and PG(n, 2) (section 5). By using the combinatorial formula in Labarthe [8] we define the co-momenta (section 6) and the full p_n-J symbol and derive our main result (section 7). We then examine examples and show how to obtain sum rules for the 3n-j coefficients (section 8). To conclude the paper we consider the interpretation of the full p_n-J symbol (section 9).

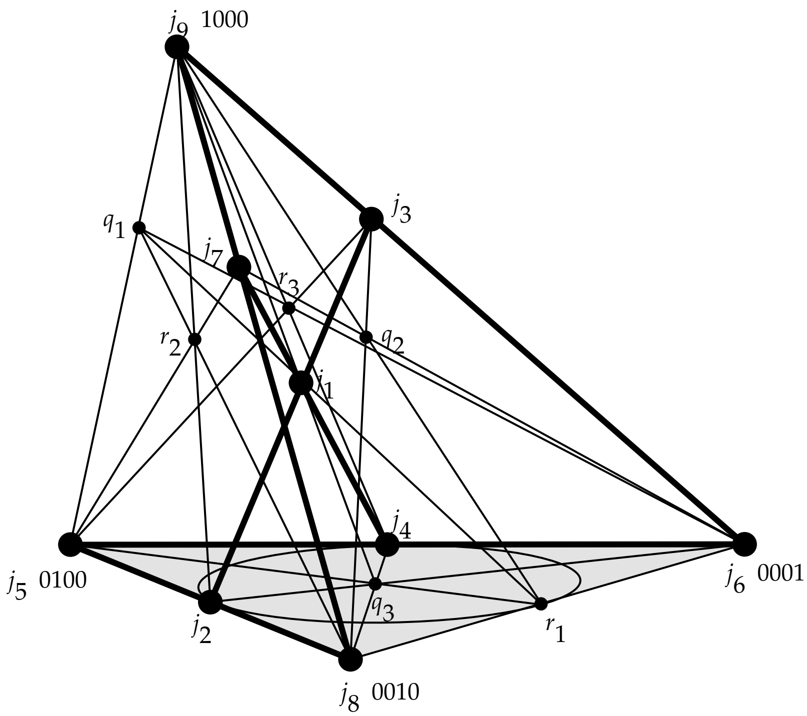

Figure 3. The PG(3, 2) geometry of the 9-j coefficient. Seven lines pass through each of the 15 points, but only some of the 35 lines are represented. One of the fifteen 2-subspaces (Fano planes) is shaded.

2. Angular momentum graphs

A $3n$-j coefficient is defined by its angular momentum graph G (see [9–12]). The graph defines an invariant of SU(2) constructed from $3n$ momenta which label the edges. G is a trivalent graph (three edges meet at each of the $2n$ vertices).

The $3n$-j coefficient ($n = 1$) in figure 4 is the θ coefficient so named from its shape. The graph defines the value of the θ coefficient as

$$\theta_{j_1 j_2 j_3} = \sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} (-)^{j_1 - m_1 + j_2 - m_2 + j_3 - m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{2}$$

The left vertex in figure 4 corresponds to a coupling of the three angular momenta j_1, j_2, j_3 and to the first 3-j symbol in equation (2). The $\pm$ sign at a vertex of G fixes the cyclic order of the momenta in the 3-j symbol in equation (2). The way the projection number m of momentum j and the phase factor appear in equation (2), as $(\cdots {}^{j}_{m} \cdots)$ and $(-)^{j-m}(\cdots {}^{j}_{-m} \cdots)$, is fixed by the arrow on edge j. Changing the sign of a vertex (j_1, j_2, j_3) or the direction of an edge j multiplies the $3n$-j by $(-)^{j_1+j_2+j_3}$ or $(-)^{2j}$ respectively. The value of $\theta_{j_1 j_2 j_3}$ is 1 when the set of momenta (j_1, j_2, j_3) satisfies the *triangular conditions*, namely

$$j_1 + j_2 - j_3 \qquad j_2 + j_3 - j_1 \qquad \text{and} \qquad j_3 + j_1 - j_2 \tag{3}$$

are non-negative integers, and 0 otherwise.

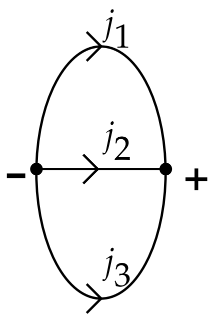

Figure 4. The θ coefficient.

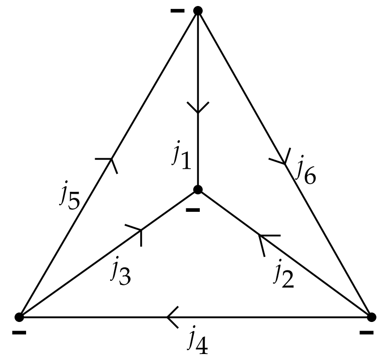

Figure 5. The 6-j coefficient.

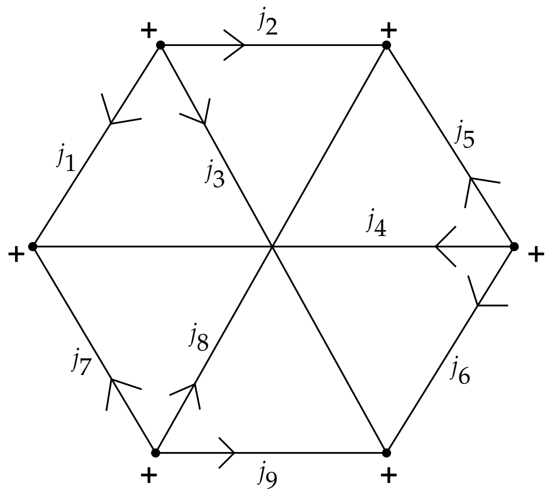

Figure 6. The 9-j coefficient.

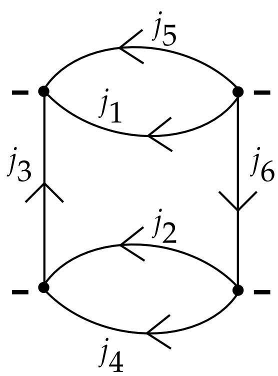

Figure 7. The 6-j' coefficient.

The value of the general $3n$-j coefficient is defined similarly by its graph. The 6-j coefficient $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ is represented by the tetrahedron graph (figure 5) and the 9-j coefficient $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$ is given by figure 6.

We shall also admit as a $3n$-j ($n = 2$) the *6-j of second kind* in figure 7, which we call 6-j' and denote

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}' = \frac{(-)^{j_1+j_2+j_4+j_5}}{2j_6+1}\theta_{j_1 j_5 j_6}\theta_{j_2 j_4 j_6}\delta_{j_3 j_6}. \tag{4}$$

3. The edge space and its cycle subspace

Let E be the set of the edges of graph G. For the 6-j coefficient the set of the edges is $E = \{j_1, j_2, \ldots, j_6\}$, where we designate the edges by the same symbol as the momenta. The *edge space* $\mathcal{E}$ of the graph is the vector space over the two-element field $\mathbb{F}_2$ of functions

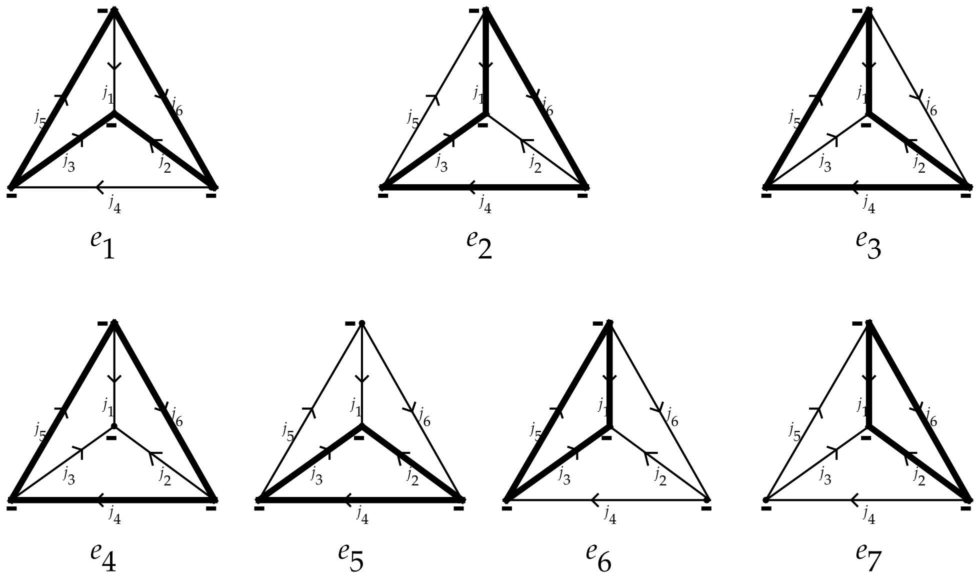

Figure 8. The non-zero cycles of the 6-j coefficient.

$E \to \mathbb{F}_2$. The support of a function $f \in \mathcal{E}$ is the subset $F \subseteq E$ of the edges $e \in E$ such that $f(e) = 1$. We shall not distinguish between a function and its support. The sum of two edge subsets $F, F' \subseteq E$ is then their symmetric difference $(F \setminus F') \cup (F' \setminus F)$.

A *cycle* of graph G is a subset of edges which form any number of *circuits* (for a formal definition see for example Diestel [13]). For the 6-j coefficient, there are $8 = 2^3$ cycles, the seven cycles represented in figure 8 and the empty set (the *zero cycle*). Figure 9 gives the seven non-zero cycles of the 6-j' of figure 7. All non-zero cycles consist of one circuit each, excepted e_3 which is formed of two circuits. Given two cycles e_1, e_2 and scalars $\lambda_1, \lambda_2 \in \mathbb{F}_2$, the linear combination $\lambda_1 e_1 + \lambda_2 e_2$ is a cycle: the cycles form the *cycle-subspace* $\mathcal{C}$ of $\mathcal{E}$. The dimension of $\mathcal{C}$ for a graph with c components, a vertices and b edges is $b - a + c$ (see Biggs [14]). For simplicity, we shall assume that the $3n$-j graph is connected ($c = 1$), but the exposition that follows remains valid with $c > 1$ components if the coefficient is thought of as a '$3(n + c - 1)$-j' instead of a $3n$-j. For example, a composite graph of two θ, though involving six angular momenta, should be considered as a '9-j'. For a $3n$-j, the cycle-subspace is of dimension $n + 1$ and contains 2^{n+1} elements.

4. The projective space PG(n^*, 2)

Let us first consider the 6-j coefficient. A basis of $\mathcal{C}$ is given by the cycles e_5, e_6, e_7. We code element $xe_5 + ye_6 + ze_7$ by the binary number xyz (for example $e_2 = e_5 + e_7$ is coded by 101). We identify the seven non-zero cycles e_i ($i = 1, 2, \ldots, 7$) of $\mathcal{C}$ with the Fano plane PG(2^*, 2) (figure 10). Similar projective spaces have been considered for general graphs in connection with the theory of graph colourings (see Holton and Sheenhan [15] and Tutte [16]).

To generalize to any $3n$-j we introduce the projective space PG(n^*, 2). The $p_n = 2^{n+1} - 1$ points of this space are identified with the non-zero cycles of the $3n$-j. The *k-subspaces* of PG(n^*, 2) are the points ($k = 0$), the lines ($k = 1$), the planes ($k = 2$), ... and

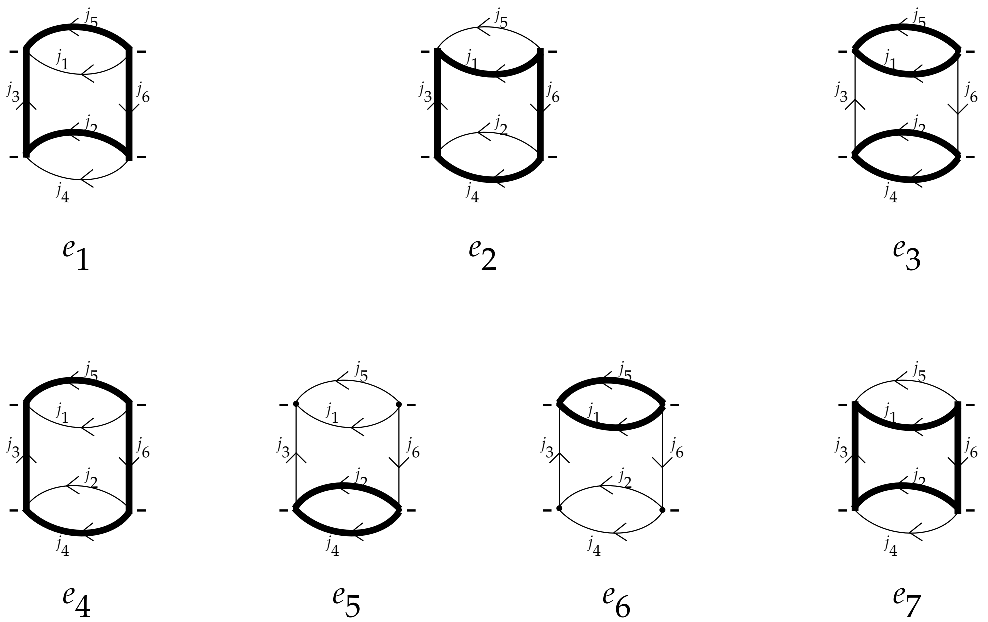

Figure 9. The non-zero cycles of the 6-j' coefficient.

the hyperplanes ($k = n - 1$). They are projective subspaces of projective dimension k, for example each of the p_n hyperplanes can be considered as a projective space PG$(n-1, 2)$ of $2^n - 1$ points.

In the evaluation of the 6-j coefficient we shall need to know the cycles that contain a given edge. For example edge j_1 of the 6-j is contained in cycles e_2, e_3, e_6 and e_7 (see figure 8): these are the points of the Fano plane (figure 10) outside the line (e_1, e_4, e_5). Let us now determine the cycles $e \in \mathcal{C}$ that contain a given edge $a \in E$ in the general case. Let $\chi_a(e)$ be the function $\mathcal{C} \to \{-1, +1\}$ defined by $\chi_a(e) = -1$ if cycle e contains edge a and $\chi_a(e) = 1$ otherwise. The function χ_a is a character (in this paper the word 'character' will mean irreducible character) of the Abelian group $\mathcal{C}$:

$$\chi_a(e + e') = \chi_a(e)\chi_a(e') \qquad \text{for } e, e' \in \mathcal{C}. \tag{5}$$

The characters of $\mathcal{C}$ are associated to the hyperplanes of PG$(n^*, 2)$: χ_a takes the value -1 on the points outside a hyperplane a^* and $+1$ on the points of the hyperplane a^* and at 0 (in $\mathcal{C}$).

5. The projective space PG(n, 2) of momenta

In the duality of projective spaces, the preceding hyperplane a^* of PG$(n^*, 2)$ is identified with a point a of PG$(n, 2)$. Figure 2 represents PG$(2, 2)$ for the 6-j. The point $\lambda\mu\nu$ of PG$(2, 2)$ corresponds to the line of PG$(2^*, 2)$ given by equation $\lambda x + \mu y + \nu z = 0$ for the points xyz in PG$(2^*, 2)$. Six points of PG$(2, 2)$ are labelled by the momenta of the 6-j which are thus *embedded* in PG$(2, 2)$. For example point 011 of PG$(2, 2)$, corresponding to line (e_1, e_4, e_5) of equation $y + z = 0$ of PG$(2^*, 2)$ is labelled with j_1. Figures 2 and 10 have been drawn so as to make their duality apparent.

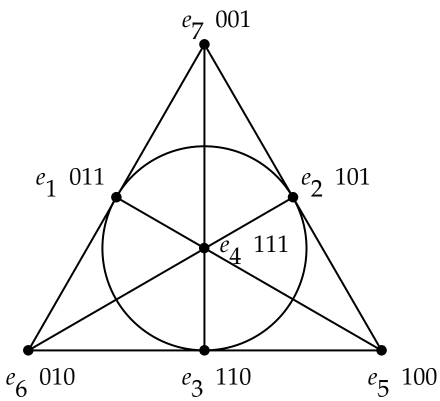

Figure 10. The Fano plane of co-momenta PG$(2^*, 2)$.

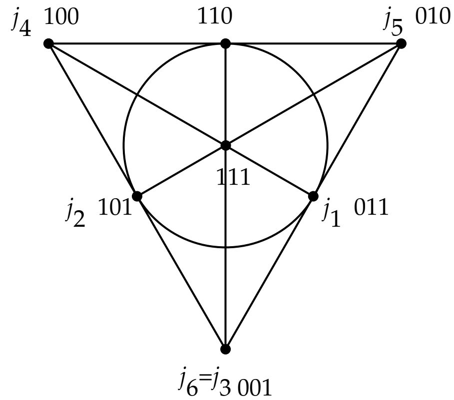

Figure 11. The Fano plane PG$(2, 2)$ of momenta for the 6-j'.

For the 6-j', we have labelled the cycles in figure 9 so that the projective space PG$(2^*, 2)$ is the same as for the 6-j (figure 10). In the labelling of PG$(2, 2)$ by the momenta of the 6-j' that results (figure 11), only five of the seven points are labelled with momenta, j_3 and j_6 labelling the same point (corresponding to the fact that the 6-j' is 0 unless $j_3 = j_6$).

When an edge a of an angular momentum graph G is an *isthmus*, meaning that one part of the graph can be removed by cutting this edge, the corresponding $3n$-j is 0 unless momentum $a = 0$. Since an isthmus is not contained in any cycle of G, the corresponding momentum is not embedded in PG$(n, 2)$.

We remark that a coupling of angular momenta is represented by a line of PG$(2, 2)$. For example (j_2, j_4, j_6) form a line. This property generalizes to PG$(n, 2)$. Indeed, let (a, b, c) be three edges coupled in the $3n$-j coefficient. In each cycle there is an even number of these edges, so that the corresponding characters are linked by

$$\chi_a(e)\chi_b(e)\chi_c(e) = 1 \qquad \forall\ e \in \mathrm{PG}(n^*, 2) \tag{6}$$

implying that the hyperplanes a^*, b^*, c^* associated to the characters have an $(n-2)$-subspace in common. In the dual space PG$(n, 2)$ this means that the corresponding points (a, b, c) are collinear.

6. The co-momenta

The combinatorial formula of the $3n$-j coefficient is based on its non-zero cycles (Labarthe [8], where the non-zero cycles are called *closed diagrams*). To each non-zero cycle e_i is associated a sign $\epsilon_i = \pm 1$ computed by the following rules:

- orient all circuits of e_i in an arbitrary fashion.
- multiply the factors:

—at each vertex of e_i, a factor of $+1$ if the cyclic order of the edges is (incoming edge, outgoing edge, third edge) and -1 otherwise (figure 12).

—on each edge of e_i, a factor of $+1$ if the directions of the edge and cycle are opposite or -1 if they are the same (figure 13).

—a factor -1 for each circuit.

Note that the value of ϵ_i is independent of the orientation chosen for the circuits. *Examples*: for the θ coefficient (figure 4) $\epsilon_i = -1$ for the three non-zero cycles; for the 6-j

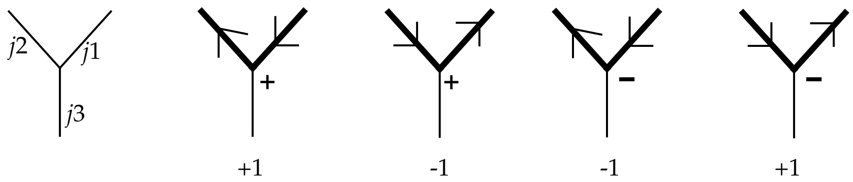

Figure 12. Vertex factor.

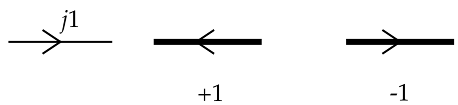

Figure 13. Edge factor.

and 6-j' coefficients (figures 5 and 7) $\epsilon_i = 1$ for all non-zero cycles.

To each set of values of momenta in a $3n$-j we associate an array x of these $3n$ values. For example, we associate the 1×3 array $x = [\, j_1 \quad j_2 \quad j_3 \,]$ to $\theta_{j_1 j_2 j_3}$ and the 2×3 array $x = \begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix}$ to the 6-j $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$. We denote by R the space of arrays like these when the entries are integers or half-integers that satisfy the triangular conditions of the $3n$-j, with $\{x\}$ the value of the $3n$-j associated to array $x \in R$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of non-negative integers.

For arrays in R, we have the usual addition and multiplication by a scalar $\lambda \in \mathbb{N}$. For example, in the case of the θ, if $x = [\, j_1 \quad j_2 \quad j_3 \,]$ and $x' = [\, j_1' \quad j_2' \quad j_3' \,]$ we have $x + x' = [\, j_1 + j_1' \quad j_2 + j_2' \quad j_3 + j_3' \,]$ and $\lambda x = [\, \lambda j_1 \quad \lambda j_2 \quad \lambda j_3 \,]$. It is easy to see that if $x, x' \in R$ and $\lambda \in \mathbb{N}$ then $x + x' \in R$, $\lambda x \in R$ (R is closed under addition and multiplication by a non-negative integer scalar).

To each non-zero cycle e_i we associate an array in R corresponding to momenta of $\frac{1}{2}$ on the edges of e_i and zero for the remaining edges. To simplify notations, these elements of R are denoted by the same name as the cycles. They are for the 6-j:

$$
\begin{aligned}
&e_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad
e_2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \qquad
e_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \\
&e_4 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad
e_5 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \qquad
e_6 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \\
&e_7 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.
\end{aligned} \tag{7}
$$

Each element $x \in R$ can be decomposed over these p_n arrays $e_i \in R$ ($i = 1, 2, \ldots, p_n$) as

$$
x = \sum_{i=1}^{p_n} l_i e_i \qquad l_i \in \mathbb{N}. \tag{8}
$$

We call l_i the *co-momentum* at point e_i of PG(n^*, 2). Since the arrays e_i are not independent in R, decomposition (8) is not unique in general, but the number of different decompositions is always finite because the co-momenta have to be non-negative integers.

The normalizing factor N of the $3n$-j coefficient is the product of the triangle factors of the $2n$ couplings (a, b, c):

$$
N = \prod_{(a,b,c)} \Delta_{abc} \tag{9}
$$

$$\Delta_{abc} = \left(\frac{(a+b-c)!(b+c-a)!(c+a-b)!}{(a+b+c+1)!}\right)^{1/2}. \tag{10}$$

The value of the $3n$-j coefficient is expressed as:

$$\{x\} = N \sum \left(\prod_{i=1}^{p_n} (-\epsilon_i)^{l_i}\right) \frac{(|l|+1)!}{\prod_{i=1}^{p_n} l_i!} \tag{11}$$

where the sum is over the decompositions (8) of x in co-momenta and where $|l| = \sum_{i=1}^{p_n} l_i$.

For a given set of co-momenta, equation (8) gives the momenta of x. For example, for the 6-j we have using equation (7):

$$\begin{aligned} j_1 &= (l_2 + l_3 + l_6 + l_7)/2 \\ j_2 &= (l_1 + l_3 + l_5 + l_7)/2 \\ j_3 &= (l_1 + l_2 + l_5 + l_6)/2 \\ j_4 &= (l_2 + l_3 + l_4 + l_5)/2 \\ j_5 &= (l_1 + l_3 + l_4 + l_6)/2 \\ j_6 &= (l_1 + l_2 + l_4 + l_7)/2. \end{aligned} \tag{12}$$

We see, more generally, that the momentum j_k associated to point k of PG$(n, 2)$ is half the sum of the co-momenta at points in PG$(n^*, 2)$ outside the hyperplane dual to k, or in terms of characters:

$$j_k = \tfrac{1}{2} \sum_{\chi_k(i)=-1} l_i \qquad \text{for } k = 1, 2, \ldots, p_n. \tag{13}$$

Let us mention that equations similar to equation (13) occur with different quantities and different meanings in Conway (assisted by Fung) [17], where PG$(2, 2)$ and its dual are used in the geometric classification of the three-dimensional lattices in terms of their Voronoï cells, and in Fairlie and Ueno [18, equations (10) and (11)], where PG$(n, 2)$ is associated with integrable top equations. We have extended equation (13) to define momenta at all points of PG$(n, 2)$, even those that were not momenta in the $3n$-j coefficient. So for the 6-j, we have added momentum j_7 at point 111 of figure 2 with value

$$j_7 = (l_4 + l_5 + l_6 + l_7)/2. \tag{14}$$

This momentum j_7 satisfies the triangular relations corresponding to lines in PG$(2, 2)$: the sets (j_1, j_4, j_7), (j_2, j_5, j_7) and (j_3, j_6, j_7) are triangular if the co-momenta are non-negative integers. This property remains true in the general setting of PG$(n, 2)$. Let (a, b, c) be a triplet of momenta on line d in PG$(n, 2)$. Since the corresponding dual hyperplanes a^*, b^*, c^*, which have the $(n-2)$-subspace d^* in common, cover all PG$(n^*, 2)$, we have, using equation (13),

$$a + b - c = \sum l_i \tag{15}$$

where the sum is over the points of c^* not in d^*. This shows that $a + b - c \in \mathbb{N}$, and similarly $b + c - a \in \mathbb{N}$ and $c + a - b \in \mathbb{N}$, meaning that the set (a, b, c) is triangular, if the co-momenta are non-negative integers.

Introducing

$$l_0 = -\sum_{i=1}^{p_n} l_i = -|l| \qquad j_0 = 0 \tag{16}$$

and denoting the character with constant value 1 on $\mathcal{C}$ by χ_0, we can rewrite equation (13) as

$$j_k = -\tfrac{1}{4}\sum_{i=0}^{p_n} \chi_k(i) l_i \qquad \text{for } k = 0, 1, 2, \ldots, p_n. \tag{17}$$

This means that the momenta are discrete Fourier transforms of the co-momenta, so that the inverse Fourier transform of equation (13) is

$$l_i = -\frac{1}{2^{n-1}}\sum_{k=1}^{p_n} \chi_k(i) j_k \qquad \text{for } i = 1, 2, \ldots, p_n \tag{18}$$

with, for the sum of co-momenta,

$$|l| = \frac{1}{2^{n-1}}\sum_{k=1}^{p_n} j_k. \tag{19}$$

For example, for the 6-j, the inverse of equations (12) and (14) is

$$\begin{aligned}
l_1 &= (-j_1 + j_2 + j_3 - j_4 + j_5 + j_6 - j_7)/2\\
l_2 &= (+j_1 - j_2 + j_3 + j_4 - j_5 + j_6 - j_7)/2\\
l_3 &= (+j_1 + j_2 - j_3 + j_4 + j_5 - j_6 - j_7)/2\\
l_4 &= (-j_1 - j_2 - j_3 + j_4 + j_5 + j_6 + j_7)/2\\
l_5 &= (-j_1 + j_2 + j_3 + j_4 - j_5 - j_6 + j_7)/2\\
l_6 &= (+j_1 - j_2 + j_3 - j_4 + j_5 - j_6 + j_7)/2\\
l_7 &= (+j_1 + j_2 - j_3 - j_4 - j_5 + j_6 + j_7)/2
\end{aligned} \tag{20}$$

with sum

$$|l| = (+j_1 + j_2 + j_3 + j_4 + j_5 + j_6 + j_7)/2. \tag{21}$$

7. The formula of hidden momenta

We now interpret equation (11) by defining a *full p_n-J symbol*. Let X be an array of p_n angular momenta j_k $(k = 1, 2, \ldots, p_n)$. We associate j_k with point k in PG$(n, 2)$ and the co-momentum l_i $(i = 1, 2, \ldots, p_n)$ calculated by equation (18) with point i in PG$(n^*, 2)$. The co-momenta are now allowed to take rational values and not only integer values. We define the value of the full p_n-J symbol $\langle X\rangle$ by

$$\langle X\rangle = \frac{(-)^{|l|}(|l|+1)!}{\prod_{i=1}^{p_n} l_i!} \tag{22}$$

if the co-momenta are non-negative integers (with $0! = 1$) and zero otherwise. Of course, $\langle X\rangle = 0$ if the momenta (a, b, c) on a line of PG$(n, 2)$ are not triangular, but, as we shall see in the next section, it can happen that some co-momenta are negative or non-integer so that $\langle X\rangle = 0$ even when all triangular conditions are satisfied. We call *visible momenta* the momenta of the full p_n-J symbol which were originally in the $3n$-j coefficient and *hidden momenta* those which were not.

Using equation (11), we arrive at our main result, *the formula of hidden momenta*. The value of the $3n$-j coefficient is given by

$$\{x\} = N \sum (-)^{t(X)} \langle X\rangle \tag{23}$$

where the sum is over the hidden momenta of the full p_n-J symbol and

$$(-)^{t(X)} = \prod_{i=1}^{p_n} (\epsilon_i)^{l_i} \tag{24}$$

is a phase factor.

The formula of hidden momenta interprets the summations in the combinatorial expression: they have the role of concealing the hidden momenta in the full p_n-J symbol, so that only the visible momenta remain.

8. Examples and sum rules

In this section we give examples for the θ, 6-j and 9-j coefficients. It will appear that among the full p_n-J symbol that satisfy all triangular conditions usually a large number are zero because the co-momenta fail to be non-negative integers. In the formula of hidden momenta for structurally different $3n$-j coefficients with the same n, a visible momentum in one coefficient can appear as hidden in another. This increases our confidence that hidden momenta have physical meaning and are not merely summation variables. This also results in sum rules, equations (28), (34), (57) and (59) below. We have found no reference for these sum rules which could also be derived from the combinatorial and graphical techniques in Jucys and Bandzaitis [9, equations (14.3) and (35.4)].

8.1. The θ coefficient

The θ coefficient $\theta_{j_1 j_2 j_3}$ corresponds to the projective line PG(1, 2). Its co-momenta are the three numbers involved in the triangular conditions (3); there is no hidden momentum. The full 3-J symbol of PG(1, 2), denoted $\langle j_1 j_2 j_3 \rangle$, is

$$\langle j_1 j_2 j_3 \rangle = \frac{(-)^{j_1+j_2+j_3}}{(\Delta_{j_1 j_2 j_3})^2} \tag{25}$$

when triangular conditions (3) are satisfied, and 0 otherwise. Equation (23) contains no summation:

$$\theta_{j_1 j_2 j_3} = (\Delta_{j_1 j_2 j_3})^2 (-)^{j_1+j_2+j_3} \langle j_1 j_2 j_3 \rangle. \tag{26}$$

The degenerate projective space PG(0, 2), consisting of only one point, can be considered to correspond to a degenerate $3n$-j (for $n = 0$) with an angular momentum graph formed of one single loop j and value $\{j\} = 2j + 1$. There is one co-momentum $l = 2j$, and equations (22)–(24) apply if we put $\langle j \rangle = (-)^{2j}(2j+1)$ for the full 1-J symbol of PG(0, 2), $\epsilon_1 = -1$ for the sign of the non-zero cycle and $N = 1$ for the normalizing factor.

The composite angular momentum graph $\{j_1\}\{j_2\}$ formed of two loops j_1 and j_2 appears as embedded in the line PG(1, 2) with a hidden momentum j_3. The formula of hidden momenta equation (23), which reads

$$\{j_1\}\{j_2\} = \sum_{j_3} (-)^{2j_1+2j_2} \langle j_1 j_2 j_3 \rangle \tag{27}$$

can be considered as a sum rule for the θ coefficient:

$$\sum_{j_3} \frac{(-)^{j_3-j_1-j_2}\theta_{j_1 j_2 j_3}}{(\Delta_{j_1 j_2 j_3})^2} = \{j_1\}\{j_2\} = (2j_1+1)(2j_2+1). \tag{28}$$

It can be remarked that, in the full 3-J symbol, j_3 is a visible momentum in equation (26) but a hidden momentum in equation (27).

8.2. The 6-j and 6-j' coefficients

Using the same display of momenta as in figure 2, we have for the full 7-J symbol of PG(2, 2)

$$\left\langle \begin{matrix} j_4 & j_3 & j_5 \\ & j_7 & \\ j_2 & & j_1 \\ & j_6 & \end{matrix} \right\rangle = \frac{(-)^{|l|}(|l|+1)!}{l_1!l_2!\ldots l_7!} \tag{29}$$

when all co-momenta (equation (20)) are non-negative integers, and 0 otherwise. The formula of hidden momenta equation (23) is for the 6-j

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \Delta_{j_1 j_2 j_3} \Delta_{j_1 j_5 j_6} \Delta_{j_2 j_4 j_6} \Delta_{j_3 j_4 j_5} \sum_{j_7} \left\langle \begin{matrix} j_4 & j_3 & j_5 \\ & j_7 & \\ j_2 & & j_1 \\ & j_6 & \end{matrix} \right\rangle \tag{30}$$

where the summation over the hidden momentum j_7 is limited by the triangular conditions $j_1 j_4 j_7$, $j_2 j_5 j_7$ and $j_3 j_6 j_7$. Moreover, from the form of equation (20), only values of j_7 varying with step of 2 will correspond to integer co-momenta and contribute to the sum. Equation (30) is equivalent to the combinatorial formula of Racah [19, equation (36)] which reads, when transcribed in our notations,

$$\begin{aligned} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} &= \Delta_{j_1 j_2 j_3} \Delta_{j_1 j_5 j_6} \Delta_{j_2 j_4 j_6} \Delta_{j_3 j_4 j_5} \sum_z \frac{1}{(j_1+j_2-j_3-z)!(j_4+j_5-j_3-z)!} \\ &\times \frac{(-)^{j_1+j_2+j_4+j_5-z}(j_1+j_2+j_4+j_5-z+1)!}{(j_1+j_5-j_6-z)!(j_2+j_4-j_6-z)!z!(j_3+j_6-j_1-j_4+z)!(j_3+j_6-j_2-j_5+z)!} \end{aligned} \tag{31}$$

where z runs over integer values such that all factorials have non-negative arguments. Indeed, equation (31) becomes algebraically identical to equation (30) when the sum over z is changed to a sum over j_7 by identifying z with co-momentum l_3, that is by putting

$$z = (j_1 + j_2 - j_3 + j_4 + j_5 - j_6 - j_7)/2. \tag{32}$$

The formula of hidden momenta gives thus an entirely new perspective on Racah formula, revealing that it is not merely a combinatorial expression.

The formula of hidden momenta for the 6-j' (with two momenta equal to j_6) is

$$\begin{Bmatrix} j_1 & j_2 & j_6 \\ j_4 & j_5 & j_6 \end{Bmatrix}' = (\Delta_{j_1 j_5 j_6} \Delta_{j_2 j_4 j_6})^2 \sum_{j_3, j_7} \left\langle \begin{matrix} j_4 & j_3 & j_5 \\ & j_7 & \\ j_2 & & j_1 \\ & j_6 & \end{matrix} \right\rangle. \tag{33}$$

Comparing equations (30), (33) and (4), we get the sum rule:

$$\begin{aligned} \sum_{j_3} \frac{1}{\Delta_{j_1 j_2 j_3} \Delta_{j_3 j_4 j_5}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} &= \frac{1}{\Delta_{j_1 j_5 j_6} \Delta_{j_2 j_4 j_6}} \begin{Bmatrix} j_1 & j_2 & j_6 \\ j_4 & j_5 & j_6 \end{Bmatrix}' \\ &= \frac{(-)^{j_1+j_2+j_4+j_5} \theta_{j_1 j_5 j_6} \theta_{j_2 j_4 j_6}}{(2j_6+1) \Delta_{j_1 j_5 j_6} \Delta_{j_2 j_4 j_6}}. \end{aligned} \tag{34}$$

Let us now examine the fact that, even when the hidden momenta satisfy all triangular conditions, the co-momenta are not necessarily non-negative integers. The 66 sets of values of hidden momenta j_3, j_7 of the 6-j'

$$\begin{Bmatrix} j_1 & j_2 & j_6 \\ j_4 & j_5 & j_6 \end{Bmatrix}' = \begin{Bmatrix} 5 & \frac{9}{2} & 4 \\ \frac{13}{2} & 6 & 4 \end{Bmatrix}' \tag{35}$$

which satisfy the triangular conditions are displayed in figure 14. The full 7-J symbol is different from zero only for the 24 sets of values indicated by full circles '●'.

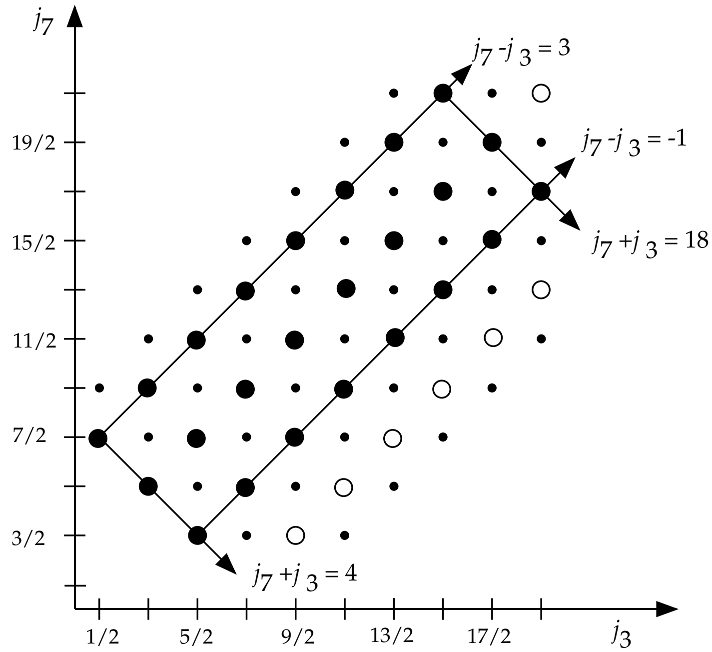

Figure 14. The values of hidden momenta j_3, j_7 of the 6-j' coefficient. Values satisfying triangular conditions are represented: the full circles ● correspond to non-negative integer co-momenta, the open circles ○ to integer co-momenta some of which are negative and small dots '·' to non-integer co-momenta. The rectangle corresponds to non-negative co-momenta.

Let $j_3^{(0)}, j_7^{(0)}$ be a particular set of hidden momenta that correspond to integer co-momenta and put

$$j_3 = j_3^{(0)} + x_1 \qquad j_7 = j_7^{(0)} + x_2. \tag{36}$$

We see from triangular conditions that the *shift vector* (x_1, x_2) has integer components and from equation (20) that shift vectors corresponding to integer co-momenta form the square lattice defined by:

$$\{(x_1, x_2)\colon\ x_1 \text{ and } x_2 \text{ are both even integers or}\\ x_1 \text{ and } x_2 \text{ are both odd integers}\}. \tag{37}$$

An example of *generator matrix* (see Conway and Sloane [20]) for the lattice is $M = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. The two row-vectors of M form a basis of the square lattice and the shift vectors are given by $(x_1, x_2) = zM$ where $z = (z_1, z_2)$ is a vector with integer components. In figure 14 this lattice (after a shift of origin) is represented by full or open circles.

Finally, let us determine on what condition the co-momenta are all non-negative. From equation (20), this is

$$\begin{aligned} |j_1 - j_2 - j_4 + j_5| + j_6 \leqslant j_7 + j_3 \leqslant j_1 + j_2 + j_4 + j_5 - j_6 \\ |j_1 + j_2 - j_4 - j_5| - j_6 \leqslant j_7 - j_3 \leqslant -|j_1 - j_2 + j_4 - j_5| + j_6 \end{aligned} \tag{38}$$

meaning that the hidden momenta belong to the rectangle in figure 14.

Table 1. Description of the projective geometry $\mathcal{P} = \mathrm{PG}(3, 2)$ of the 9-j coefficient.

			j_6	j_8	r_1	j_5	j_4	j_2	q_3	j_9	j_3	j_7	q_2	q_1	r_3	r_2	j_1
		ϵ	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
l_1	0001	$-$	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
l_2	0010	$-$	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
l_3	0011	$+$	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
l_4	0100	$-$	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
l_5	0101	$-$	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0
l_6	0110	$-$	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
l_7	0111	$-$	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1
l_8	1000	$-$	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
l_9	1001	$-$	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0
l_{10}	1010	$-$	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
l_{11}	1011	$-$	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1
l_{12}	1100	$-$	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
l_{13}	1101	$+$	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1
l_{14}	1110	$+$	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1
l_{15}	1111	$-$	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0

8.3. The 9-j coefficient

The description of the projective geometry $\mathcal{P} = \mathrm{PG}(3, 2)$ (and its dual $\mathcal{P}^* = \mathrm{PG}(3^*, 2)$) of the 9-$j$ coefficient is summarized in table 1. The rows (resp. columns) of the table are associated with points of $\mathcal{P}^*$ (resp. $\mathcal{P}$). The entry a_{ik} in row $i = xyzt$ and column $k = \lambda\mu\nu\rho$ has value $\lambda x + \mu y + \nu z + \rho t \in \mathbb{F}_2$.

The assignment of momenta (the nine visible momenta $j_1, j_2, \ldots, j_9$ of the 9-j and the six hidden momenta q_1, q_2, q_3, r_1, r_2, r_3) to points of $\mathcal{P}$ is the same as in figure 3.

The cycles of the 9-j are given by the rows of the table. For example, the first row represents the point 0001 of $\mathcal{P}^*$ identified with the cycle (j_6, j_4, j_3, j_1) formed of momenta corresponding to entries of 1 (disregarding the hidden momenta r_1, q_3, q_2 and r_3).

The hyperplanes (Fano planes) of $\mathcal{P}^*$ are given by the columns of the table. For example, the first column represents the point $\lambda\mu\nu\rho = 0001$ of $\mathcal{P}$ and the hyperplane of equation $\lambda x + \mu y + \nu z + \rho t = 0$ formed of points 0010, 0100, 0110, 1000, 1010, 1100 and 1110 of $\mathcal{P}^*$ corresponding to zero entries in the table.

Similarly, the hyperplanes of $\mathcal{P}$ are given by the rows of the table. For example, the points of the shaded hyperplane in figure 3 correspond to the zero entries of row 1000 of the table.

The characters are given by $\chi_k(i) = -1$ (resp. $\chi_k(i) = 1$) when entry $a_{ik} = 1$ (resp. $a_{ik} = 0$).

The expression of the momenta in terms of co-momenta, equation (13), is read from the columns of the table. For example, the first column gives

$$j_6 = (l_1 + l_3 + l_5 + l_7 + l_9 + l_{11} + l_{13} + l_{15})/2 \tag{39}$$

where each term corresponds to an entry of 1 in the table.

The expression of the co-momenta in terms of momenta, equation (18), is read from the rows of the table. For example, the first row gives

$$l_1 = (+j_6 - j_8 + r_1 - j_5 + j_4 - j_2 + q_3 - j_9 + j_3 - j_7 + q_2 - q_1 + r_3 - r_2 + j_1)/4 \tag{40}$$

where a $+$ (resp. $-$) sign corresponds to an entry of 1 (resp. 0) in the table.

The sign of the cycles is $\epsilon_i = 1$ for the three collinear cycles forming line $d = (0011, 1101, 1110)$ of $\mathcal{P}^*$ and $\epsilon_i = -1$ for the other 12 non-zero cycles. The phase factor $(-)^{t(X)}$ equation (24) can be expressed in terms of the momenta: the sum of co-momenta l_i with $\epsilon_i = -1$

$$t(X) = q_1 + q_2 + q_3 \tag{41}$$

turns out to be the sum of the momenta on the dual line of d, a result that can be understood geometrically by the same method as for equation (15).

8.3.1. The 35 lines. Let us give some geometric properties of the lines of $\mathcal{P}$. The set $\mathcal{H}$ of the nine points $j_1, j_2, \ldots, j_9$ of $\mathcal{P}$ is the set of points that satisfy the quadratic equation $\lambda\mu = \nu\rho$. The figure $\mathcal{H}$ representing the 9-j is a hyperboloid. The 35 lines of $\mathcal{P}$ can be classified with respect to $\mathcal{H}$ as follows.

• Six are the *interior* lines which are contained in $\mathcal{H}$ and correspond to the triangular conditions of the 9-j (these lines are also considered to be *tangents* of $\mathcal{H}$):

$$\begin{array}{lll} j_1j_2j_3 & j_4j_5j_6 & j_7j_8j_9 \\ j_1j_4j_7 & j_2j_5j_8 & j_3j_6j_9. \end{array} \tag{42}$$

• Nine are *tangents* of $\mathcal{H}$ that have just one point in common with $\mathcal{H}$:

$$\begin{array}{lll} j_1q_1r_1 & j_2q_2r_3 & j_3q_3r_2 \\ j_4q_2r_2 & j_5q_3r_1 & j_6q_1r_3 \\ j_7q_3r_3 & j_8q_1r_2 & j_9q_2r_1. \end{array} \tag{43}$$

• 18 are the *secants* of $\mathcal{H}$, having exactly two points in common with $\mathcal{H}$:

$$\begin{array}{llllll} j_2j_7q_1 & j_1j_5q_2 & j_1j_9q_3 & j_2j_4r_1 & j_1j_6r_2 & j_1j_8r_3 \\ j_3j_4q_1 & j_3j_8q_2 & j_2j_6q_3 & j_3j_7r_1 & j_2j_9r_2 & j_3j_5r_3 \\ j_5j_9q_1 & j_6j_7q_2 & j_4j_8q_3 & j_6j_8r_1 & j_5j_7r_2 & j_4j_9r_3. \end{array} \tag{44}$$

• Two are the *exterior* lines of $\mathcal{H}$, having no point in common with $\mathcal{H}$:

$$q_1q_2q_3 \qquad r_1r_2r_3. \tag{45}$$

Through each point of $\mathcal{H}$ there are three tangents (among which two interior lines) and four secants. The *tangent space* at a point of $\mathcal{H}$ is the Fano plane containing the tangents through that point. For example, the shaded hyperplane in figure 3 is the tangent space at point j_5.

Through each point of $\mathcal{P} \setminus \mathcal{H}$ there are three secants, three tangents and one exterior line.

8.3.2. The formula of hidden momenta for the 9-j. We have, for the full 15-J symbol,

$$\left\langle \begin{matrix} j_1 & j_2 & j_3 & & & \\ j_4 & j_5 & j_6 & q_1 & q_2 & q_3 \\ j_7 & j_8 & j_9 & r_1 & r_2 & r_3 \end{matrix} \right\rangle = \frac{(-)^{|l|}(|l|+1)!}{l_1!l_2!\ldots l_{15}!} \tag{46}$$

when all co-momenta (given by equations like equation (40)) are non-negative integers, and 0 otherwise.

The formula of hidden momenta for the 9-j coefficient is the six-fold summation formula:

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{matrix} \right\} = N \sum_{\substack{q_1q_2q_3 \\ r_1r_2r_3}} (-)^{q_1+q_2+q_3} \left\langle \begin{matrix} j_1 & j_2 & j_3 & & & \\ j_4 & j_5 & j_6 & q_1 & q_2 & q_3 \\ j_7 & j_8 & j_9 & r_1 & r_2 & r_3 \end{matrix} \right\rangle \tag{47}$$

with the normalizing factor

$$N = \Delta_{j_1 j_2 j_3}\Delta_{j_4 j_5 j_6}\Delta_{j_7 j_8 j_9}\Delta_{j_1 j_4 j_7}\Delta_{j_2 j_5 j_8}\Delta_{j_3 j_6 j_9} \tag{48}$$

and summation limited to hidden momenta satisfying the 29 triangular conditions in equations (43)–(45). In this formula we have used equation (41) in the expression of the phase factor, but, since the co-momenta have to be integers, we can replace $(-)^{q_1+q_2+q_3}$ by equivalent forms, such as $(-)^{t'(X)}$ with

$$t'(X) = q_1 + q_2 + q_3 - 2l_1 + 2l_{13} = j_2 - j_3 - j_4 + j_5 + j_7 + j_9 + q_1. \tag{49}$$

As in the case of the 6-j', there are hidden momenta that correspond to non-integer co-momenta, even when all triangular conditions equations (43)–(45) are satisfied. We define as before the shift vector $(x_1, x_2, \ldots, x_6)$ with integer components by

$$\begin{aligned} q_1 &= q_1^{(0)} + x_1 \qquad q_2 = q_2^{(0)} + x_2 \qquad q_3 = q_3^{(0)} + x_3 \\ r_1 &= r_1^{(0)} + x_4 \qquad r_2 = r_2^{(0)} + x_5 \qquad r_3 = r_3^{(0)} + x_6 \end{aligned} \tag{50}$$

where $q_1^{(0)}, \ldots, r_3^{(0)}$ is a set of hidden momenta that corresponds to integer co-momenta. Shift vectors corresponding to integer co-momenta form the lattice D_6^+ defined by:

$$\Big\{(x_1, x_2, \ldots, x_6)\colon \text{ all } x_i \text{ even integers or all } x_i \text{ odd integers}, \quad \sum_{i=1}^{6} x_i \equiv 0 \pmod 4\Big\}. \tag{51}$$

An example of generator matrix for the lattice is

$$M = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \end{pmatrix}. \tag{52}$$

The shift vectors are given by $(x_1, x_2, \ldots, x_6) = zM$ where $z = (z_1, z_2, \ldots, z_6)$ is a vector with integer components. A fundamental cell of the lattice, containing only one shift vector, has volume $\det M = 64$. It results that usually in equation (47) a large fraction of the summed terms are zero because the corresponding co-momenta are non-integers. The hidden momenta are further limited to the polytope in six dimensions (generalizing the rectangle in figure 14) defined by $l_i \geqslant 0$ ($i = 1, 2, \ldots, 15$). For example, for the 9-j coefficient $\begin{Bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{Bmatrix}$ there are 25 564 sets of hidden momenta that satisfy all triangular conditions, 426 of which correspond to integer co-momenta. Moreover, among these 426 sets only eight sets correspond to non-negative co-momenta and so to non-zero terms in the summation of equation (47).

Various six-fold summation formulae for the 9-j have been given by Wu [21]. They are based on a generating function for the 9-j, first obtained by Schwinger [22] in a creation and annihilation operator approach, and then rederived by Wu using a formalism from Bargmann [23]. The combinatorial formula equation (11) of the $3n$-j coefficient used in this paper derives from a generating function which for the 9-j is the same as that of Schwinger. It results that these various six-fold summation formulae are equivalent to equation (47) in the sense that the non-zero terms that are summed are the same in all formulae. Here again,

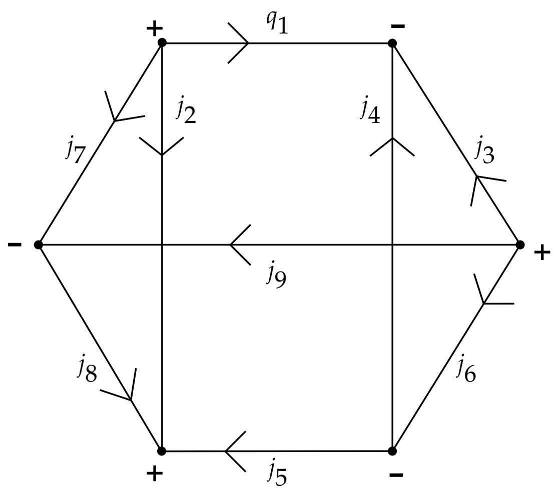

Figure 15. The 9-j' coefficient.

as in the case of Racah formula, the formula of hidden momenta adds physical meaning to these combinatorial expressions.

8.3.3. The 9-j coefficient of the second kind. The *9-j′ coefficient* (9-j coefficient of the second kind) defined by figure 15 is a product of two 6-j coefficients:

$$\begin{Bmatrix} q_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}' = \begin{Bmatrix} j_2 & j_7 & q_1 \\ j_9 & j_5 & j_8 \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & q_1 \\ j_5 & j_9 & j_6 \end{Bmatrix}. \tag{53}$$

We have labelled the momenta so that this 9-j' is embedded in $\mathcal{P}$ as in figure 3 and table 1, but now the visible momenta are $q_1, j_2, j_3, \ldots, j_9$ and the hidden momenta are $j_1, q_2, q_3, r_1, r_2, r_3$, so that momenta j_1 and q_1 have swapped the visible/hidden attributes they had in the 9-j. The full 15-J symbol is still given by equation (46) and the cycles by table 1. For example, point 0001 of $\mathcal{P}^*$ is associated with cycle (j_6, j_4, j_3). The signs of the cycles are now $\epsilon_i = 1$ for the three collinear cycles $i = 0001, 0010, 0011$ and $\epsilon_i = -1$ for the other 12 non-zero cycles. The phase factor $(-)^{t(X)}$ is independent of the hidden momenta:

$$t(X) = q_1 + j_5 + j_9. \tag{54}$$

The formula of hidden momenta for the 9-j' coefficient is the six-fold summation formula:

$$\begin{Bmatrix} q_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}' = N'(-)^{q_1+j_5+j_9} \sum_{\substack{j_1 q_2 q_3 \\ r_1 r_2 r_3}} \left\langle \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{matrix} \begin{matrix} q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{matrix} \right\rangle \tag{55}$$

with the normalizing factor

$$N' = \Delta_{j_2 j_7 q_1} \Delta_{j_3 j_4 q_1} \Delta_{j_2 j_5 j_8} \Delta_{j_3 j_6 j_9} \Delta_{j_4 j_5 j_6} \Delta_{j_7 j_8 j_9}. \tag{56}$$

Comparing with equation (47), used with expression equation (49) instead of $q_1+q_2+q_3$ in the phase factor, we get the sum rule:

$$\sum_{j_1} \frac{1}{\Delta_{j_1 j_2 j_3} \Delta_{j_1 j_4 j_7}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$$

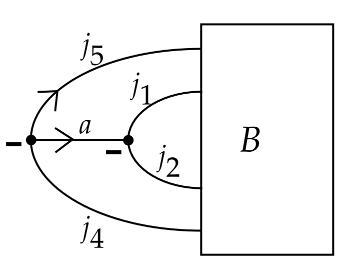

Figure 16. The graph G_a of the $3n$-j coefficient $\{U, a\}$.

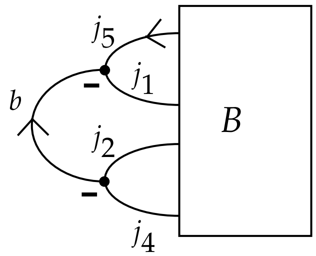

Figure 17. The graph G_b of the $3n$-j coefficient $\{U, b\}$.

$$= (-)^{j_2-j_3-j_4+j_7} \sum_{q_1} \frac{1}{\Delta_{j_2 j_7 q_1} \Delta_{j_3 j_4 q_1}} \begin{Bmatrix} j_2 & j_7 & q_1 \\ j_9 & j_5 & j_8 \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & q_1 \\ j_5 & j_9 & j_6 \end{Bmatrix}. \tag{57}$$

8.4. A sum rule

The sum rules, equations (28), (34) and (57), can be generalized to a sum rule relating the $3n$-j coefficients depicted in figures 16 and 17. The graphs G_a and G_b have an identical part (box B and edges j_5, j_1, j_2, j_4) excepted that orientations of edge j_5 are opposite. They generalize figures 5 and 7 by replacing edge j_6 and its two adjacent vertices by box B and renaming j_3 as a or b. We denote the $3n$-j by $\{U, a\}$ and $\{U, b\}$ where U is the set of common angular momenta. Both $3n$-j can be embedded in the same projective space $\mathrm{PG}(n, 2)$. Let us first suppose that no two momenta are associated to the same point of $\mathrm{PG}(n, 2)$. We have then a full p_n-J symbol $\langle U, a, b, H\rangle$ where visible momentum a of $\{U, a\}$ also appears as a hidden momentum of $\{U, b\}$ and where we denote by H the set of the common hidden momenta.

The cycles of the two graphs are completely defined by their edges on U. It is easily checked that the sign of a cycle of G_a is the same as the sign of the cycle of G_b that has the same edges on U. (Some of the cases to be considered occur in figures 8 and 9, where cycles e_i ($i = 1, \ldots, 7$) with the same i have the same edges on $U = \{j_1, j_2, j_4, j_5, j_6\}$.) The $3n$-j are thus given by the formula of hidden momenta as

$$\begin{aligned} \{U, a\} &= \Delta_{a j_1 j_2} \Delta_{a j_4 j_5} N_B \sum_{H,b} (-)^t \langle U, a, b, H\rangle \\ \{U, b\} &= \Delta_{b j_1 j_5} \Delta_{b j_2 j_4} N_B \sum_{H,a} (-)^t \langle U, a, b, H\rangle \end{aligned} \tag{58}$$

with the same phase factor and where N_B is a product of common triangle factors. From these equations we obtain the sum rule relating the $3n$-j coefficients in figures 16 and 17:

$$\sum_a \frac{1}{\Delta_{a j_1 j_2} \Delta_{a j_4 j_5}} \{U, a\} = \sum_b \frac{1}{\Delta_{b j_1 j_5} \Delta_{b j_2 j_4}} \{U, b\}. \tag{59}$$

This sum rule remains valid when momentum b is associated to the same point of $\mathrm{PG}(n, 2)$ as another momentum (as in figure 11): the sum on b is then reduced to one term only as in equation (34).

9. Concluding remarks

The full p_n-J symbol (22) is highly symmetric. It is invariant in the $p_n!$ permutations of the co-momenta. In the space of momenta, these symmetries form a group of linear transformations. A subgroup of these symmetries consists of the collineations, which permute points and lines, both in PG$(n, 2)$ and its dual, thus conserving the set of values of the momenta as well as the set of their triangular relations.

For the 6-j, among the $7! = 5040$ symmetries, 144 leave j_7 fixed at the same point in PG$(2, 2)$: these are the Regge [24] symmetries of the 6-j. Among these, 6×4 are collineations that correspond to the 24 geometric symmetries of the tetrahedron (figure 5).

We give a state model interpretation of the full p_n-J symbol (22) and of the co-momenta inspired by the chromatic method of evaluating Penrose spin networks (see Penrose [25], Moussouris [26] and Kauffman [27]). Let us consider a system of $|l|$ identical objects that are partitioned in p_n boxes, box i containing l_i objects ($i = 1, 2, \ldots, p_n$). The co-momenta l_i thus appear as occupation numbers. We have available $c \geqslant |l|$ colours to paint the objects. We call a *state* of the system a colouring of the $|l|$ objects such that no two objects have the same colour. The number of states of this system is

$$P(X, c) = \frac{c(c-1)\ldots(c-|l|+1)}{\prod_{i=1}^{p_n} l_i!}. \tag{60}$$

Considered in terms of variable c for given X, equation (60) defines a polynomial that takes integer values when c is a positive or negative integer. Evaluating the polynomial at $c = -2$, which corresponds to the 'number of colours' of spin networks, yields the full p_n-J symbol (22).

For θ, 6-j and 9-j coefficients, the chromatic method of evaluating spin networks, based on *circuit* decompositions gives a formula similar to equation (11), based on *cycle* decompositions. The two formulae differ for the other $3n$-j coefficients, when a cycle can be composed of several circuits as in figure 9. Instead of the simple expression (60) for a cycle configuration, the chromatic polynomial for a circuit configuration has a more complex expression. This becomes important when one considers the generalization of the $3n$-j to quantum group q-$3n$-j coefficients. The principle of replacing factorials $k!$ by quantum integer factorials $[k]!$ is useful for guessing the quantum formula of the q-$3n$-j. Kauffman and Lins [28, pp 91–2] pointed out that the quantization of the chromatic polynomial to higher q-$3n$-j was not clear. With our method of cycle decompositions, equation (22) has the same structure for all $3n$-j, so that we have a way to quantize the general formula of the $3n$-j.

Let us rewrite diagram (1) as

$$\boxed{\begin{array}{c}\text{co-momentum}\\ \text{object}\end{array}} \quad \overset{\text{duality}}{\longleftrightarrow} \quad \boxed{\begin{array}{c}\boxed{\begin{array}{c}\text{angular-momentum}\\ \text{object}\end{array}}\\ \boxed{\begin{array}{c}\text{hidden part of}\\ \text{angular-momentum}\\ \text{object}\end{array}}\end{array}} \tag{61}$$

We have seen the validity of such a diagram when the angular-momentum object is a $3n$-j. When the angular-momentum object is a spin-j elementary particle state $|jm\rangle$, it makes sense to suppose that there is no hidden part and that the co-momenta are $\lambda = j + m$, $\mu = j - m$, usually interpreted as occupation numbers of two-oscillator states $|\lambda\mu\rangle$. We can imagine that, for a system of elementary particles, the hidden part of diagram (61) appears after couplings and recouplings of the single-particle states. The formula in Racah [19, equation (16)] for the 3-j coefficient (Clebsch–Gordan) can be interpreted as a summation over a hidden projection angular momentum, but we have not yet obtained a clear interpretation for the general angular momentum graph with free edges (projection angular momenta). Another open question is what these hidden structures are, if they exist, for atomic and nuclear shells and their matrix elements.

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